Stabilizing near-nonhyperbolic chaotic systems and its potential applications in neuroscience

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Based on the invariance principle of differential equations a simple, systematic, and rigorous feedback scheme with the variable feedback strength is proposed to stabilize nonlinearly any chaotic systems without any prior analytical knowledge of the systems. Especially the method may be used to control near-nonhyperbolic chaotic systems, which although arising naturally from models in astrophysics to those for neurobiology, all OGY-type methods will fail to stabilize. The technique is successfully used to the famous Hindmarsh-Rose model neuron and the Rössler hyperchaos system.

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Since Ott, Grebogi, and York (OGY) [1] proposed an effective method of chaos control, all kinds of variations based on this method have been given [2], and lots of successful experiments have been reported. For the simplicity, here we call the OGY method and its variations as the OGY-type methods. Recall the idea of the OGY-type methods, the following three steps are necessary to improve its performance: (i) To specify and locate an unstable periodic orbit embedded in the chaotic attractor, say a fixed point x_f; (ii) To approximate linearly the system in a small neighborhood of x_f by reconstructing statistically the corresponding linearized matrix J; (iii) To control (or stabilize) the chaotic orbits entering the neighborhood to x_f with aid of the approximated linear dynamics. The first step can be realized by the method of close returns from experimental data, and the third step is entirely within the field of linear control theory such as "the pole placement technique". Although the second step, including the calculation of eigenvalues and eigenvectors of the corresponding linearized matrix J, has been solved by the least-squares fit, this problem is related to how chaos affects the linear estimation of the dynamics in the small neighborhood of x_f. Especially, when the system is nonherpybolic and any prior analytical knowledge of dynamics is not available, such linearization will be problematic due to the

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nonlinearity. It is well known that many of the chaotic phenomena seen in systems occurring in practice are nonhyperbolic. On the other hand, there are numerous successful reports of OGY control in numerical experiments. This matter is slightly puzzling. In [3], the author investigated carefully this problem, and found that there are two possible reasons resulting in such contradiction. One reason is that the least-squares fit used in the process of reconstructing the attractor from time series is ill-defined due to the nonhyperbolicity of system. The other is that there are large relative errors in the process of solving numerically eigenvalues of a matrix as one of its eigenvalues, $\lambda \approx 0$, which is a well-known fact in the matrix computations [4]. Therefore in those successful numerical experiments the nonhyperbolicity of system may be destroyed before obtaining the information for attempting the OGY control by experimental time series. Although the nonhyperbolicity of the chaotic attractor does not automatically mean the nonhyperbolicity of the unstable periodic orbits embedded in it, the near-nonhyperbolicity must exist and affect the performance of the OGY control when the system itself is near-nonhyperbolic, see the models discussed below. More interestingly, the report in [5], on failure of chaos control in a parametrically excited pendulum whose excitation mechanism is not perfect, throws highly the light to this viewpoint (to the best of my knowledge, this is the first report on failure of the OGY control in the concrete physical experiments).

This letter is motivated by the limitation of the OGY-type controllers as what referred above. Especially we address the control problem on the near-nonhyperbolic chaotic systems in the form of

$$\dot{u} = g_u(u, v), \qquad \dot{v} = r g_v(u, v), \tag{1}$$

where $u \in \mathrm{R}^{n_1}$, $v \in \mathrm{R}^{n_2}$, and $0 < r \ll 1$. The systems have simultaneously n_1 dramatic components u and n_2 slow variables v, which arise naturally from many scientific disciplines, and range from models in astrophysics to those for biological cells [6]. In particular, such systems and their discrete versions are widely used to model bursting, spiking, and chaotic phenomena in neuroscience, see [7] and references therein. More interestingly, just as what referred in [1], due to multipurpose flexibility of higher life form, chaos may be a necessary ingredient in their regulation by the brain. In [3] the author guessed that such chaotic ingredient is probably in the form of (1), where the slow variables represent a "container" or"recorder" storing the acquired knowledge, which is attached on some neurons. The guess is mainly based on a fact of cognitive science, namely, the acquisition of some knowledge will give way to acquire other knowledge in the brain, and hence the acquisition of knowledge will "decrease" the freedom of topology structure of the brain [8]. If the chaotic ingredient in the brain is from the systems in the form of (1), then all OGY-type methods will fail to control such chaotic dynamics because the systems with sufficiently small r are near-nonhyperbolic (actually note that in discrete case if the linearization matrix J admits one eigenvalue $\lambda \approx 1$ (resp. $\lambda \approx 0$ for the continuous case) all OGY-type controllers are infeasible because those controllers contain the near-singular term $(J - I)^{-1}$, see [2]). This is just the reason for failure of chaos control reported in [5]. In the meantime, this reflects the fact that the slow varies, i.e., the neurons recording the knowledge, should not be ignored for achieving

multipurpose flexibility of the brain because the related knowledge has to be excited to respond to the signals entering the brain. Moreover, this mechanism is beneficial to explain why stabilization of an inverted triple pendulum is very troublesome as out-of planar motions become very substantial, which was firstly reported in [9]. In this letter, based on the invariance principle of differential equations [10], a simple and rigorous feedback scheme with the variable feedback strength is proposed to stabilize nonlinearly any chaotic and hyperchaotic systems without any prior analytical knowledge of the systems. Especially, this simple technique can be easily applied to stabilize near-nonhyperbolic chaotic systems in the form of (1). This letter is mainly focused on the continuous systems, but the proposed method can be generalized to the case of the discrete version by the invariance principle of difference equations.

Let a chaotic system be given as

$$\dot{x} = f(x), \tag{2}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathrm{R}^n$, $f(x) = (f_1(x), f_2(x), \cdots, f_n(x)) : \mathrm{R}^n \to \mathrm{R}^n$ is a nonlinear vector function. Without loss of the generality we let $\Omega \subset \mathrm{R}^n$ be a chaotic bounded set of (1) which is globally attractive, and suppose that $x = 0$ is a fixed point embedded in Ω. For the vector function $f(x)$, we give a general assumption.

For any $x = (x_1, x_2, \cdots, x_n) \in \Omega$, there exists a constant $l > 0$ satisfying

$$\mid f_i(x) \mid \leq l \max_j \mid x_j \mid, i = 1, 2, \cdots, n. \tag{3}$$

Note this condition is very loose, for example, the condition (3) holds as long as $\frac{\partial f_i}{\partial x_j}(i, j = 1, 2, \cdots, n)$ are bounded. Therefore the class of systems in the form of (2)-(3) include almost all well-known chaotic and hyperchaotic systems. To stabilize the chaotic orbits in (2) to the fixed point $x = 0$, we consider the feedback control

$$\dot{x} = f(x) + \epsilon x. \tag{4}$$

Instead of the usual linear feedback, the feedback strength $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ here will be duly adapted according to the following update law:

$$\dot{\epsilon}_i = -\gamma_i x_i^2, i = 1, 2, \cdots, n, \tag{5}$$

where $\gamma_i > 0, i = 1, 2, \cdots, n$, are arbitrary constants. For the system consisting of (4) and (5), we introduce the following function

$$V = \frac{1}{2}\sum_{i=1}^{n} x_i^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\gamma_i}(\epsilon_i + L)^2, \tag{6}$$

where L is a constant bigger than nl, i.e., $L > nl$. By differentiating the function V along the trajectories of the system (4)-(5), we obtain

$$\dot{V} = \sum_{i=1}^{n} x_i(f_i(x) + \epsilon_i x_i) - \sum_{i=1}^{n}(\epsilon_i + L)x_i^2 \leq (nl - L)\sum_{i=1}^{n} x_i^2 \leq 0. \tag{7}$$

where we have assumed $x \in \Omega$ (without loss of the generality as Ω is globally attractive), and used the condition (3). It is obvious that $\dot{V} = 0$ if and only if $x = 0$, namely the set $E = \{(x, \epsilon) \in \mathrm{R}^{2n} : x = 0, \epsilon = \epsilon_0 \in \mathrm{R}^n\}$ is the largest invariant set contained in $\dot{V} = 0$ for the system (4)-(5). Then according to the well-known invariance principle of differential equations [10], starting with arbitrary initial values of the system (4)-(5), the orbit converges asymptotically to the set E, i.e., $x \to 0$ and $\epsilon \to \epsilon_0$ as $t \to \infty$.

Namely, when the chaotic system (2) is stabilized to $x = 0$ the variable feedback strength ϵ will be automatically adapted to a suitable strength ϵ_0 depending on the initial values. This is significantly different from the usual linear feedback, and the converged strength must be of the lower order than those used in the constant gain schemes. But theoretically the converged strength may be very big so that it may give rise to its own dynamics. However the flexibility of the strength in the present scheme can overcome this limitation once such case arises. For example, suppose that the feedback strength is restricted not to exceed a critical value, say k. In the present control procedure, once the variable strength ϵ exceeds k at time $t = t_0$, we may choose the values of variables at this time as initial values and repeat the same control by resetting the initial strength $\epsilon(0) = 0$. Namely one may achieve the stabilization within the restricted feedback strength due to the global stability of the present scheme. This idea is slightly similar to that of the OGY control [1], i.e., small parameter control, but there exists a certain difference, for example, in the OGY control the controller waits passively for the emergence of chaotic orbits. In the other side, in the present scheme the small converged strength may be obtained by adjusting suitably the parameter γ. Moreover, we note that in the present scheme it is not necessary for some particular models to use all the variables of the system as feedback signals. For example, one may set $\epsilon_i \equiv 0$ if $\mid e_i \mid \leq \mid e_j \mid$, and this case exists in general due to the nonhyperbolicity of chaotic attractor, see the following examples. Obviously this simple, systematic, and rigorous method may stabilize nonlinearly almost all chaotic systems without any priori analytical knowledge of systems, and is robust against the effect of noise due to the global nonlinear stability.

Next we will give two illustrative examples. We think that the chaotic ingredient in the brain is from the systems in the form of (1), and thus we take the famous Hindmarsh-Rose model neuron [11] as the first example, which is governed by the following three-order ordinary differential equation

$$\dot{x}_1 = x_2 + 3x_1^2 - x_1^3 - x_3 + I, \quad \dot{x}_2 = 1 - 5x_1^2 - x_2, \quad \dot{x}_3 = -rx_3 + 4r(x_1 + 1.6) \quad (8)$$

with $0 < r \ll 1$. Here x_1 is the membrane potential of the neuron, x_2 is a recovery variable and x_3 is a slow adaptation current. It has been found in [12] that the model admits a chaotic attractor with $r = 0.0012$ and the external current $I = 3.281$, see Figure 1 for the chaotic time series of x_1. After transforming only one fixed point $(-0.6835, -1.3359, 3.666)$ to $(0, 0, 0)$, we stabilize successfully this near-nonhyperbolic chaotic system by the proposed scheme, where let $\epsilon_2 \equiv 0$. The corresponding numerical results and the evolution of ϵ are shown in Figure 2, where the initial values are set as $(-0.5, -0.3, 0.1, 0, 0)$ with $\gamma_1 = 0.01, \gamma_3 = 0.1$. In

addition, in Figure 3 we show that this chaotic system may also be stabilized by another two feedback signals x_2 and x_3, where all initial values are same those in Figure 2 and parametrical values are set as $\gamma_2 = 0.01, \gamma_3 = 0.1$. However we find numerically that such stabilization is troublesome by the feedback signals x_1 and x_2, which confirms our viewpoint referred above, namely, it is difficult to stabilize the near-nonhyperbolic chaotic systems if this near-nonhyperbolicity (resp. the slow variable) is ignored.

To show the generality of the present method, our second example is the famous Rössler hyperchaos system:

$$\dot{x}_1 = -x_2 - x_3, \quad \dot{x}_2 = x_1 + 0.25x_2 + x_4, \quad \dot{x}_3 = 3 + x_1 x_3, \quad \dot{x}_4 = -0.5x_3 + 0.05x_4. \tag{9}$$

Accordingly after transforming only one fixed point $(-5.4083, -0.5547, 0.5547, 5.547)$ to $(0, 0, 0, 0)$, the hyperbolic chaotic system is stabilized by the present method, where let $\epsilon_1 = \epsilon_3 \equiv 0$. The corresponding numerical results and the evolution of ϵ are shown in Figure 4, where the initial values are set as $(5, 30, 5, 10, 0, 0)$ with $\gamma_2 = \gamma_4 = 0.2$.

In conclusion, we have given a simple, systematic, and rigorous method to stabilize nonlinearly any chaotic systems. Especially the method may be used to near-nonhyperbolic chaotic system, which all OGY-type methods will fail to stabilize. Perhaps this idea may explain more reasonably the multipurpose flexibility of higher life form due to chaotic ingredient of the brain. In addition, this idea of control has been applied successfully to chaotic synchronization by the author [13], so we believe that the idea may be used to explore the interesting dynamical properties found in neurobiological systems, i.e., the onset of regular bursts in a group of irregularly bursting neurons with different individual properties [14].

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References

[1] E. Ott, C. Grebogi and J.A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990); T. Shinbrot, *et al*, Nature **363**, 411 (1993).

[2] L. Yang, Z. Liu and J. Mao, Phys. Rev. Lett. **84**, 67 (2000); J. Mao, Z. Liu and L. Yang, Phys. Rev. E **62**, 4846 (2000); V.A. Oliveira and J.R. Rosolen, Phys. Rev. E **68**, 058201 (2003).

[3] Debin Huang, Chin. Phys. Lett. **19**, 762 (2002).

[4] G.W. Stewart, *Introduction to matrix computations*, New York: Academic Press, (1973).

[5] W. van de Water and J. de Weger, Phys. Rev. E **62**, 6398 (2000).

[6] E.A. Spiegel, Cosmic arrhythmias, *Chaos in Astrophysics* ed J.R. Buchler *et al*, Dortrecht: Reidel, 91 (1985); T. Chay and J. Rinzel, Biohys. J. **47**, 357 (1985).

[7] M.I. Rabinovich and H.D.I. Abarbanel, Neuroscience, **87**, 5 (1998); E.M. Izhikevich, Int. J. Bifur. Chaos **10**, 1171 (2000); SIAM Rev. **43**, 315 (2001).

[8] In other words, the neuron attaching the knowledge is little active in comparision with other neurons except when the related signals enter the brain, thus it will not die out and its dynamics varies slowly. Linking with the models of neurobiology, such guess should be reasonable.

[9] D.J. Acheson and T. Mullin, Nature **366**, 215 (1993).

[10] J.P. Lasalle, Pro. Natl. Acad. Sci. **46**, 363 (1960); IRE Trans. Circuit Theory **7**, 520 (1960).

[11] J.L. Hindmarsh and R.M. Rose, Proc. R. Soc. Lond. B **221**, 87 (1984).

[12] X.J. Wang, Physica D **62**, 263 (1993); Y.S. Fan and T.R. Chay, Phys. Rev. E **51**, 1012 (1995); R. Huerta, *et al*, Phys. Rev. E **55**, R2108 (1997); M. Bazhenov, *et al*, Physica D **116**, 392 (1998).

[13] Debin Huang, Phys. Rev. E **69**, (5) (2004); Chaos **14**, 152 (2004); A uniform controller for synchronizing identical chaotic systems, submitted to Phys. Rev. Lett., (2004).

[14] N.F. Rulkov, Phys. Rev. Lett. **86**, 183 (2001); M.G. Rosenblum and A.S. Pikovsky, Phys. Rev. Lett. **92**, 114102 (2004).

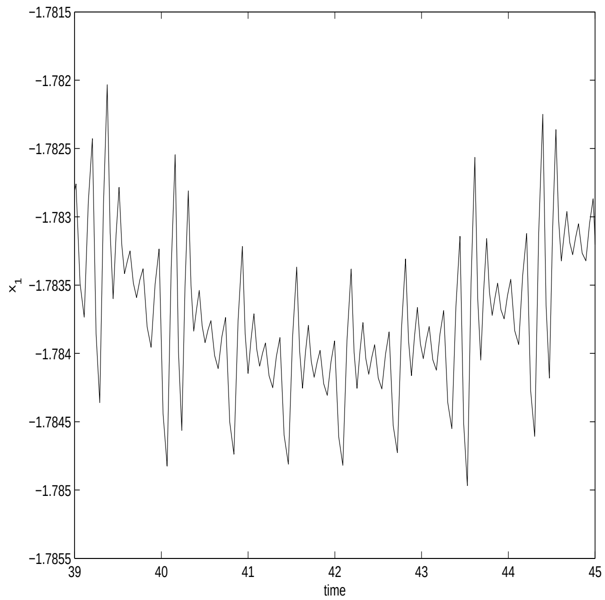

FIG.1. Time series $x_1(t)$ generated by the chaotic Hindmarsh-Rose model (8).

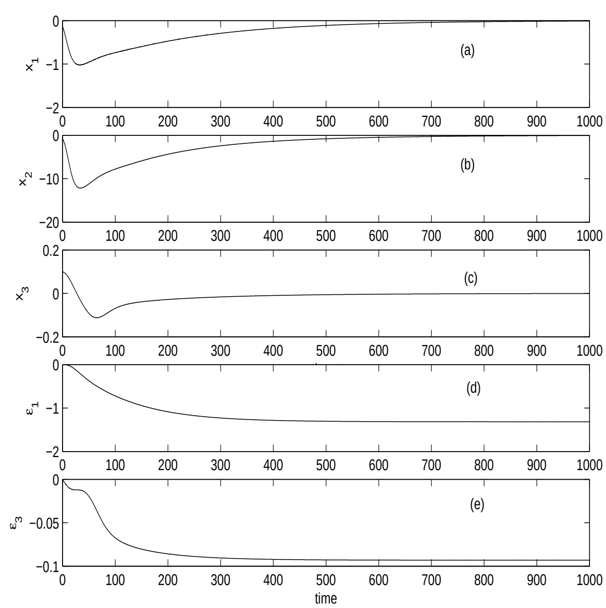

FIG.2. The chaotic Hindmarsh-Rose model (8) is stabilized successfully by only two feedback signals x_1 and x_3, where (a)-(c) show the temporal evolution of the variables $x_i, i = 1, 2, 3$, and (d)-(e) correspond to the variable feedback strength ϵ_1 and ϵ_3.

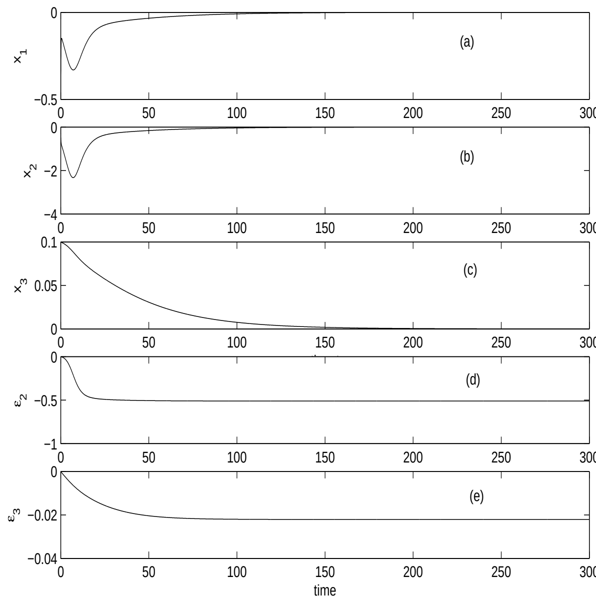

FIG.3. The chaotic Hindmarsh-Rose model (8) may also be stabilized by another two feedback signals x_2 and x_3.

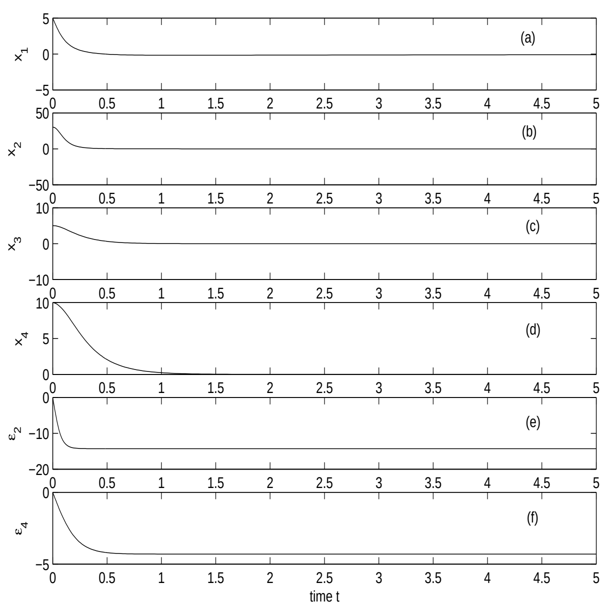

FIG.4. The Rössler hyperchaotic system (9) is stabilized by only two feedback signals x_2 and x_4, where (a)-(d) show the temporal evolution of the variables $x_i, i = 1, 2, 3, 4$, and (e)-(f) correspond to the temporal evolution of feedback strength ϵ_2 and ϵ_4.